

SMALL COVER AND HALPERIN-CARLSSON CONJECTURE - II

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ABSTRACT. For a small cover Q^n and any principal $(\mathbb{Z}_2)^m$ -bundle M^n over Q^n , it was shown in [2] that the total sum of \mathbb{Z}_2 -Betti numbers of M^n is at least 2^m . In this paper, we prove that when M^n is connected, the total sum of \mathbb{Z}_2 -Betti numbers of such an M^n exactly equals 2^m if and only if M^n is homeomorphic to a product of spheres, and Q^n in this case must be a generalized real Bott manifold (or equivalent, Q^n is a small cover over a product of simplices).

1. INTRODUCTION

let \mathbb{Z}_2 denote the quotient (additive) group $\mathbb{Z}/2\mathbb{Z}$. Based on some basic construction of principal $(\mathbb{Z}_2)^m$ -bundles over smooth manifolds introduced in [1], the following theorem is proved in [2].

Theorem 1.1 (Yu [2]). *If $(\mathbb{Z}_2)^m$ acts freely on a manifold M^n whose orbit space is a small cover, we must have:*

$$\sum_{i=0}^{\infty} \dim_{\mathbb{Z}_2} H^i(M^n, \mathbb{Z}_2) \geq 2^m. \quad (1)$$

This provides some new evidence to support the Halperin-Carlsson conjecture for free $(\mathbb{Z}_2)^m$ -actions which claims that if $(\mathbb{Z}_2)^m$ can act freely on a finite CW-complex X , we should have

$$\sum_{i=0}^{\infty} \dim_{\mathbb{Z}_2} H^i(X, \mathbb{Z}_2) \geq 2^m \quad (2)$$

The reader is referred to [3]— [8] for more information about the Halperin-Carlsson conjecture. In particular, it is interesting see what kind of free $(\mathbb{Z}_2)^m$ -actions and X can make the equality in (2) hold. For the sake of brevity, we

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introduce the following notation.

$$\mathrm{hrk}(X, \mathbb{Z}_2) := \sum_{i=0}^{\infty} \dim_{\mathbb{Z}_2} H^i(X, \mathbb{Z}_2).$$

We call $\mathrm{hrk}(X, \mathbb{Z}_2)$ the *total \mathbb{Z}_2 -cohomology rank* of X . In this paper, we will think of a space X with a free $(\mathbb{Z}_2)^m$ -action as a principal $(\mathbb{Z}_2)^m$ -bundle over some base space. The main result of this paper is stated as following.

Theorem 1.2. *For a small cover Q^n , there exists some principal $(\mathbb{Z}_2)^m$ -bundle M^n over Q^n with the total \mathbb{Z}_2 -cohomology rank $\mathrm{hrk}(M^n, \mathbb{Z}_2) = 2^m$ if and only if Q^n is a small cover over a product of simplices.*

Recall that an n -dimensional *small cover* is a closed n -manifold with a locally standard $(\mathbb{Z}_2)^n$ -action whose orbit space can be identified with a simple polytope (see [11]).

The most obvious examples of Q^n that satisfy the conditions in Theorem 1.2 are products of real projective spaces. But in general, Q^n could be the total space B_m of an iterated real projective space bundle as following:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where each B_i ($1 \leq i \leq m$) is the projectivization of the Whitney sum of a finite collection of real line bundles over B_{i-1} . The B_m is called a *generalized real Bott manifold* in [9]. In fact, the Remark 6.5 in [9] told us the following.

Proposition 1.3 (Choi, Masuda and Suh [9]). *The set of all generalized real Bott manifolds are exactly the set of all small covers over products of simplices.*

Corollary 1.4. *A small cover Q^n satisfies the condition in Theorem 1.2 if and only if Q^n is a generalized real Bott manifold.*

Moreover, we can prove the following.

Theorem 1.5. *Suppose M^n is a connected manifold. Then M^n is a principal $(\mathbb{Z}_2)^m$ -bundle over some small cover with $\mathrm{hrk}(M^n, \mathbb{Z}_2) = 2^m$ if and only if M^n is homeomorphic to a product of spheres.*

Obviously, if M^n is homeomorphic to a product of spheres $S^{n_1} \times \cdots \times S^{n_k}$, then the product action of the antipodal map of each S^{n_i} defines a free $(\mathbb{Z}_2)^k$ action on M^n whose orbit space is $\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}$. And in fact, $\mathrm{hrk}(M^n, \mathbb{Z}_2) = 2^k$ and $\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}$ is a small cover. This proves the sufficiency part of Theorem 1.5. But the necessity part of Theorem 1.5 is not trivial (see Section 3).

Remark 1.6. It is not necessarily that a closed connected manifold W^n with a free $(\mathbb{Z}_2)^m$ -action and $\text{hrk}(W^n, \mathbb{Z}_2) = 2^m$ must be a product of spheres. For example, Let $UT(S^{2n})$ be the unit tangent bundle of the $2n$ -dimensional sphere S^{2n} ($n \geq 1$). Then $UT(S^{2n})$ is a $(4n - 1)$ -dimensional closed manifold. If we think of S^{2n} as the unit sphere centered at the origin in \mathbb{R}^{2n+1} , then the tangent space of S^{2n} at any point can be thought of as a vector subspace of \mathbb{R}^{2n+1} . Under this viewpoint, we can represent any element in $UT(S^{2n})$ by (x, v) where $x \in S^{2n}$ and $v \in T_x(S^{2n})$ is a unit tangent vector at x . We can define two free involutions σ_1, σ_2 on $UT(S^{2n})$ by:

- $\sigma_1(x, v) = (-x, -v), \forall (x, v) \in UT(S^{2n});$
- $\sigma_2(x, v) = (x, -v), \forall (x, v) \in UT(S^{2n}).$

Obviously, $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$, so we get a free $(\mathbb{Z}_2)^2$ -action on $UT(S^{2n})$. It is not hard to see that $H^i(UT(S^{2n}), \mathbb{Z}_2) \cong H^i(S^{2n} \times S^{2n-1}, \mathbb{Z}_2)$ for all i . So we have $\text{hrk}(UT(S^{2n}), \mathbb{Z}_2) = 4 = 2^2$. But $UT(S^{2n})$ is not homeomorphic to $S^{2n} \times S^{2n-1}$ since their \mathbb{Z}_2 -cohomology ring structures are different and their rational homology groups are not isomorphic either. This example is informed to the author by M. Masuda. By our Theorem 1.5, the orbit space $UT(S^{2n})/(\mathbb{Z}_2)^2$ is not homeomorphic to any small cover.

In this example, $UT(S^{2n})$ has the same \mathbb{Z}_2 -cohomology groups as a product of spheres, so it is interesting to ask the following question.

Question: does there exists a closed connected manifold W^n with a free $(\mathbb{Z}_2)^m$ -action so that (i) $\text{hrk}(W^n, \mathbb{Z}_2) = 2^m$ and (ii) the \mathbb{Z}_2 -cohomology groups of W^n do not agree with the \mathbb{Z}_2 -cohomology groups of any product of spheres?

The paper is organized as follows. In section 2, we will review some basic definitions and results introduced in [1] and [2]. Then in section 3, we will prove Theorem 1.2 and Theorem 1.5. In particular, the “only if” part of Theorem 1.2 uses an interesting result of Choi [10] on the structure of simple polytopes.

2. SOME BACKGROUNDS AND KNOWN RESULTS

Suppose Q^n is an arbitrary n -dimensional closed connected smooth manifold. Let $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. It is well known that we can choose some embedded $(n-1)$ -dimensional submanifolds $\Sigma_1, \dots, \Sigma_k$ whose homology classes form a linear basis of $H_{n-1}(Q^n, \mathbb{Z}_2)$. If we cut Q^n open along $\Sigma_1, \dots, \Sigma_k$, i.e. we remove a small tubular neighborhood $N(\Sigma_i)$ of each Σ_i and remove the interior of each $N(\Sigma_i)$ from Q^n , we will get a nice manifold with corners $V^n = Q^n - \bigcup_i \text{int}(N(\Sigma_i))$, which is called a \mathbb{Z}_2 -core of Q^n (see [1] for the details of the construction). the

boundary of V^n is the union of some compact subsets P_1, \dots, P_k , called *panels*, that satisfy the following three conditions:

- (a) each panel P_i is a disjoint union of facets of V^n and each facet is contained in exactly one panel;
- (b) there exists a free involution τ_i on each P_i which sends a face $f \subset P_i$ to a face $f' \subset P_i$ (it is possible that $f' = f$);
- (c) for $\forall i \neq j$, $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$ and $\tau_i \circ \tau_j = \tau_j \circ \tau_i : P_i \cap P_j \rightarrow P_i \cap P_j$.

The $\{\tau_i : P_i \rightarrow P_i\}_{1 \leq i \leq k}$ is called an *involutive panel structure* on V^n (see [1] for the details of the construction of τ_i).

It was shown in [1] that any principal $(\mathbb{Z}_2)^m$ -bundle M^n over Q^n determines a map $\lambda : \{P_1, \dots, P_k\} \rightarrow (\mathbb{Z}_2)^m$ which is called a $(\mathbb{Z}_2)^m$ -coloring of V^n . And we can recover M^n from (V^n, λ) in the following way called *glue-back construction*.

$$M^n \cong M(V^n, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim \quad (3)$$

Where $(x, g) \sim (x', g')$ whenever $x' = \tau_i(x)$ for some P_i and $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$. So if x is in the relative interior of $P_{i_1} \cap \dots \cap P_{i_s}$, $(x, g) \sim (x', g')$ if and only if $(x', g') = (\tau_{i_s}^{\varepsilon_s} \circ \dots \circ \tau_{i_1}^{\varepsilon_1}(x), g + \varepsilon_1 \lambda(P_1) + \dots + \varepsilon_s \lambda(P_s))$ where $\varepsilon_j = 0$ or 1 for any $1 \leq j \leq s$ and $\tau_{i_j}^0 := id$.

Let $\theta_\lambda : V^n \times (\mathbb{Z}_2)^m \rightarrow M(V^n, \lambda)$ be the quotient map. There is a natural free $(\mathbb{Z}_2)^m$ -action on $M(V^n, \lambda)$ defined by:

$$g' \cdot \theta_\lambda(x, g) := \theta_\lambda(x, g' + g), \quad \forall x \in V^n, \quad \forall g, g' \in (\mathbb{Z}_2)^m. \quad (4)$$

And the homeomorphism between M^n and $M(V^n, \lambda)$ is equivariant with respect to the free $(\mathbb{Z}_2)^m$ -action. So we can represent any principal $(\mathbb{Z}_2)^m$ -bundle over Q^n by $M(V^n, \lambda)$ for some $(\mathbb{Z}_2)^m$ -coloring λ of V^n . Let

$\text{Col}_m(V^n) :=$ the set of all $(\mathbb{Z}_2)^m$ -colorings of V^n

$L_\lambda :=$ the subgroup of $(\mathbb{Z}_2)^m$ generated by $\{\lambda(P_1), \dots, \lambda(P_k)\}$,

$\text{rank}(\lambda) := \dim_{\mathbb{Z}_2} L_\lambda$.

Obviously, $\text{rank}(\lambda) \leq k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. If $\text{rank}(\lambda) = k$, we call λ *maximally independent* (in this case, we must have $m \geq k$).

Lemma 2.1 (Theorem 2.3 in [1]). *For any $(\mathbb{Z}_2)^m$ -coloring λ of V^n , $M(V^n, \lambda)$ has $2^{m-\text{rank}(\lambda)}$ connected components which are pairwise homeomorphic, and each connected component of $M(V^n, \lambda)$ is a principal $(\mathbb{Z}_2)^{\text{rank}(\lambda)}$ bundle over Q^n .*

Lemma 2.2 (Lemma 2.8 in [2]). *Suppose $\lambda_{\max} \in \text{Col}_k(V^n)$ is a maximally independent $(\mathbb{Z}_2)^k$ -coloring on V^n , where $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. Then for any $\lambda \in \text{Col}_k(V^n)$, $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2)$.*

Next, we review some basic facts on the small cover and real moment-angle manifold. Suppose P^n is an n -dimensional simple polytope with $k + n$ facets ($k \geq 1$). Here, simple means that each vertex of P^n is incident to exactly n facets of P^n . Let F_1, \dots, F_{k+n} be all the facets of P^n . For any $m \geq 1$, a map $\{F_1, \dots, F_{k+n}\} \rightarrow (\mathbb{Z}_2)^m$ is called a $(\mathbb{Z}_2)^m$ -coloring of P^n .

Suppose Q^n is a small cover over P^n . Then Q^n determines a $(\mathbb{Z}_2)^n$ -coloring μ of P^n that satisfies: whenever $F_{i_1} \cap \dots \cap F_{i_s} \neq \emptyset$, $\mu(F_{i_1}), \dots, \mu(F_{i_s})$ are linearly independent vectors in $(\mathbb{Z}_2)^n$. The μ is also called the *characteristic function* of Q^n (see [11]). For any face $\mathbf{f} = F_{i_1} \cap \dots \cap F_{i_l}$ of P^n , let $G_{\mathbf{f}}^\mu$ be the rank- l subgroup of $(\mathbb{Z}_2)^n$ generated by $\mu(F_1), \dots, \mu(F_l)$. Then we can recover Q^n from (P^n, μ) by:

$$Q^n = P^n \times (\mathbb{Z}_2)^n / \sim, \quad (p, w) \sim (p', w') \iff p = p', w - w' \in G_{\mathbf{f}(p)}^\mu, \quad (5)$$

where $\mathbf{f}(p)$ is the unique face of P^n that contains p in its relative interior. Let $\zeta_\mu : P^n \times (\mathbb{Z}_2)^n \rightarrow Q^n$ be the corresponding quotient map. Then the locally standard $(\mathbb{Z}_2)^n$ -action on Q^n can be written as:

$$w' \cdot \zeta_\mu(p, w) = \zeta_\mu(p, w' + w), \quad \forall p \in P^n, w, w' \in (\mathbb{Z}_2)^n \quad (6)$$

Obviously, the orbit space of this action can be identified with P^n . It was shown in [11] that the \mathbb{Z}_2 -Betti numbers of Q^n are decided only by the h -vector of P^n . In particular, $H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$. Moreover, any facet F_i of P^n is a simple polytope of dimension $n - 1$, and $\zeta_\mu(F_i \times (\mathbb{Z}_2)^n)$ is a small cover over F_i whose characteristic function μ_{F_i} on F_i is induced from μ by: $\mu_{F_i}(F_j \cap F_i) := \mu(F_j)$ for any face $F_j \cap F_i$ of F_i .

In addition, let $\{e_1, \dots, e_{k+n}\}$ be a basis of $(\mathbb{Z}_2)^{k+n}$ and define a $(\mathbb{Z}_2)^{k+n}$ -coloring μ_0 of P^n by $\mu_0(F_i) := e_i$, $1 \leq i \leq k + n$. Then the *real moment-angle manifold* $\mathbb{R}\mathcal{Z}_{P^n}$ is obtained by gluing 2^{k+n} copies of P^n together according to μ_0 and the rule in (5). Let $\Theta : P^n \times (\mathbb{Z}_2)^{k+n} \rightarrow \mathbb{R}\mathcal{Z}_{P^n}$ be the corresponding quotient map. There is a *canonical* $(\mathbb{Z}_2)^{k+n}$ -action on $\mathbb{R}\mathcal{Z}_{P^n}$ defined by:

$$g' \circledast \Theta(p, g) = \Theta(p, g' + g), \quad \forall p \in P^n, \forall g, g' \in (\mathbb{Z}_2)^{k+n}. \quad (7)$$

For the small cover Q^n , there exists a subtorus H of $(\mathbb{Z}_2)^{k+n}$ with rank k so that:

- (i) H acts freely on $\mathbb{R}\mathcal{Z}_{P^n}$ through the canonical action \circledast , and
- (ii) the orbit space $\mathbb{R}\mathcal{Z}_{P^n}/H$ is homeomorphic to Q^n .

But we remark that the subtorus $H \subset (\mathbb{Z}_2)^{k+n}$ which satisfies (i) and (ii) is not unique (see [2]).

3. PROOF OF THEOREM 1.2 AND THEOREM 1.5

Proof of Theorem 1.2. First, suppose Q^n is a small cover over a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ where $n_1 + \cdots + n_r = n$. For the sake of brevity, we denote the simple polytope $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ by Δ^I where $I = (n_1, \dots, n_r)$. It is easy to see that the number of facets of Δ^I equals $r + n_1 + \cdots + n_r = r + n$, and $\mathbb{R}\mathcal{Z}_{\Delta^I} \cong S^{n_1} \times \cdots \times S^{n_r}$. By the discussion at the end of the previous section, there exists some subtorus $H \subset (\mathbb{Z}_2)^{r+n}$ with rank r so that H acts freely on $\mathbb{R}\mathcal{Z}_{\Delta^I}$ through the canonical action, and the orbit space $\mathbb{R}\mathcal{Z}_{\Delta^I}/H$ is homeomorphic to Q^n . In other words, $S^{n_1} \times \cdots \times S^{n_r}$ is a principal $(\mathbb{Z}_2)^r$ -bundle over Q^n . Notice that $\text{hrk}(S^{n_1} \times \cdots \times S^{n_r}, \mathbb{Z}_2) = 2^r$, so we have proved the “if” part of the Theorem 1.2.

Conversely, if Q^n is a small cover over a simple polytope P^n . Let F_1, \dots, F_{k+n} be all the facets of P^n and μ be the $(\mathbb{Z}_2)^n$ -coloring (characteristic function) of P^n corresponding to Q^n . Let $\pi_\mu : Q^n \rightarrow P^n$ be the orbit map of the locally standard $(\mathbb{Z}_2)^n$ -action (see (6)). Now, assume that there exists a positive integer m and a principal $(\mathbb{Z}_2)^m$ -bundle $\xi : M^n \rightarrow Q^n$ with $\text{hrk}(M^n, \mathbb{Z}_2) = 2^m$. We want to show that P^n must be a product of simplices.

When $n = 1$, this is obviously true.

When $n = 2$, notice that the Euler characteristics of M^2 and Q^2 have the relation: $\chi(M^2) = 2^m \cdot \chi(Q^2)$. Without loss of generality, we can assume M^2 is connected (if M^2 is not connected, we just consider any one of its components). Then $\text{hrk}(M^2, \mathbb{Z}_2) = 4 - \chi(M^2)$. The assumption $\text{hrk}(M^2, \mathbb{Z}_2) = 2^m$ implies that: $2^m(\chi(Q^2) + 1) = 4$, which will force $\chi(Q^2) = 0$ or 1 . Then Q^2 must be a torus, a Klein bottle or a real projective plane. The torus and Klein bottle are small covers over the square (product of 1-simplices) and the real projective plane is the small cover over the 2-simplex. So in any case, P^2 is a product of simplices.

When $n \geq 3$, we claim the following.

Claim: any 2-dimensional face of P^n ($n \geq 3$) is either a triangle or a square.

To prove this claim, we will use the glue-back construction to analyze the principal $(\mathbb{Z}_2)^m$ -bundle M^n as we did in the proof of Theorem 1.1 in [2]. First, we can construct some special \mathbb{Z}_2 -core of Q^n in the following way. Take an arbitrary vertex v_0 of P^n and assume that F_{i_1}, \dots, F_{i_k} are those facets of P^n which are not incident to v_0 . Then according to [11], $\dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2) = k$ and the homology classes of the embedded submanifolds $\pi_\mu^{-1}(F_{i_1}), \dots, \pi_\mu^{-1}(F_{i_k})$ (called *facial submanifolds* of Q^n) form a \mathbb{Z}_2 -linear basis of $H_{n-1}(Q^n, \mathbb{Z}_2)$. Cutting Q^n open along $\pi_\mu^{-1}(F_{i_1}), \dots, \pi_\mu^{-1}(F_{i_k})$ will give us a \mathbb{Z}_2 -core V^n of Q^n .

Then our principal $(\mathbb{Z}_2)^m$ -bundle M^n over Q^n is (equivariantly) homeomorphic to $M(V^n, \lambda)$ for some $(\mathbb{Z}_2)^m$ -coloring λ on V^n . So $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) = 2^m$.

Case 1: If $m \leq k$, let $\iota : (\mathbb{Z}_2)^m \hookrightarrow (\mathbb{Z}_2)^k$ be the standard inclusion and define $\widehat{\lambda} := \iota \circ \lambda$. We consider $\widehat{\lambda}$ as a $(\mathbb{Z}_2)^k$ -coloring on V^n . The Theorem 2.1 implies that $M(V^n, \widehat{\lambda})$ consists of 2^{k-m} copies of $M(V^n, \lambda)$, so $\text{hrk}(M(V^n, \widehat{\lambda}), \mathbb{Z}_2) = 2^k$.

Case 2: If $m > k$, since $\text{rank}(\lambda) \leq k$, with a proper change of basis, we can assume $L_\lambda \subset (\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^m$. Let $\varrho : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^k$ be the standard projection. Define $\overline{\lambda} := \varrho \circ \lambda$. Similarly, we consider $\overline{\lambda}$ as a $(\mathbb{Z}_2)^k$ -coloring on V^n and so by Theorem 2.1, $M(V^n, \lambda)$ consists of 2^{m-k} copies of $M(V^n, \overline{\lambda})$, so $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) = 2^k$.

So in whatever case, there always exists an element $\lambda^* \in \text{Col}_k(V^n)$ so that $\text{hrk}(M(V^n, \lambda^*), \mathbb{Z}_2) = 2^k$. Moreover, by Theorem 1.1 and Lemma 2.2, we can assume that λ^* is maximally independent, i.e. $\text{rank}(\lambda^*) = k$.

Let $\xi_{\lambda^*} : M(V^n, \lambda^*) \rightarrow Q^n$ be the orbit map of the natural $(\mathbb{Z}_2)^k$ -action on $M(V^n, \lambda^*)$ defined by (4). In the proof of Theorem 1.1 in [2], it was shown that for any facet F_i of P^n , $\text{hrk}(M(V^n, \lambda^*), \mathbb{Z}_2) \geq \text{hrk}(\xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i)), \mathbb{Z}_2)$. Notice that:

- $\pi_\mu^{-1}(F_i)$ is a small cover over F_i ;
- $\xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i))$ is a principal $(\mathbb{Z}_2)^k$ -bundle over $\pi_\mu^{-1}(F_i)$.

So by Theorem 1.1, we have $\text{hrk}(\xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i)), \mathbb{Z}_2) \geq 2^k$. But by our construction, $\text{hrk}(M(V^n, \lambda^*), \mathbb{Z}_2) = 2^k$, so we must have $\text{hrk}(\xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i)), \mathbb{Z}_2) = 2^k$. Let $Y_i = \xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i))$. So Y_i is a principal $(\mathbb{Z}_2)^k$ -bundle over the small cover $\pi_\mu^{-1}(F_i)$ with $\text{hrk}(Y_i, \mathbb{Z}_2) = 2^k$ (see the following diagram).

$$\begin{array}{ccc}
 Y_i := \xi_{\lambda^*}^{-1}(\pi_\mu^{-1}(F_i)) & \xrightarrow{\subset} & M(V^n, \lambda^*) \\
 \downarrow \xi_{\lambda^*} & & \downarrow \xi_{\lambda^*} \\
 \pi_\mu^{-1}(F_i) & \xrightarrow{\subset} & Q^n \\
 \downarrow \pi_\mu & & \downarrow \pi_\mu \\
 F_i & \xrightarrow{\subset} & P^n
 \end{array}$$

By applying the above argument to the principal $(\mathbb{Z}_2)^k$ -bundle Y_i over the small cover $\pi_\mu^{-1}(F_i)$, we can show that for any codimension two face $F_i \cap F_j$ of P^n , there exists some positive integer k' and some principal $(\mathbb{Z}_2)^{k'}$ -bundle Y_{ij} over the small cover $\pi_\mu^{-1}(F_i \cap F_j)$ with $\text{hrk}(Y_{ij}, \mathbb{Z}_2) = 2^{k'}$.

Then by iterating this argument, we can show that for any 2-dimensional face \mathbf{f} of P^n , there exists some positive integer $k_{\mathbf{f}}$ and some principal $(\mathbb{Z}_2)^{k_{\mathbf{f}}}$ -bundle

$Y_{\mathbf{f}}$ over the small cover $\pi_{\mu}^{-1}(\mathbf{f})$ with $\text{hrk}(Y_{\mathbf{f}}, \mathbb{Z}_2) = 2^{k_{\mathbf{f}}}$. Then our discussion on dimension two cases suggests that \mathbf{f} must be a square or a triangle. So the claim is proved.

Then our theorem follows from the Theorem 3.1 below, which is an unpublished result of Suyoung Choi [10]. \square

Theorem 3.1 (Choi [10]). *For an n -dimensional simple polytope P^n with $n \geq 3$, then P^n is a product of simplices if and only if any 2-dimensional face of P^n is either a triangle or a square.*

Proof of Theorem 1.5. By Theorem 1.2, if M^n is a principal $(\mathbb{Z}_2)^m$ -bundle over a small cover Q^n with $\text{hrk}(M^n, \mathbb{Z}_2) = 2^m$, Q^n must be a small cover over a product of simplices $\Delta^I = \Delta^{n_1} \times \cdots \times \Delta^{n_r}$ where $n_1 + \cdots + n_r = n$. Let $\{v_0^i, \dots, v_{n_i}^i\}$ be the set of vertices of Δ^{n_i} . Then each vertex of Δ^I can be written as a product of vertices of Δ^{n_i} 's for $i = 1, \dots, r$. Hence the set of vertices of Δ^I is:

$$\{v_{j_1 \dots j_r} = v_{j_1}^1 \times \cdots \times v_{j_r}^r \mid 0 \leq j_i \leq n_i, i = 1, \dots, r\}.$$

Each facet of Δ^I is the product of a codimension-one face of Δ^{n_i} 's and the remaining simplices. So the set of facets of Δ^I is:

$$\mathcal{F}(\Delta^I) = \{F_{k_i}^i \mid 0 \leq k_i \leq n_i, i = 1, \dots, r\},$$

where $F_{k_i}^i = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f_{k_i}^i \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_r}$, and $f_{k_i}^i$ is the codimension-one face of the simplex Δ^{n_i} which is opposite to the vertex $v_{k_i}^i$. So there are $r + n$ facets in Δ^I . Since Δ^I is simple, exactly n facets meet at each vertex. Indeed, the vertex $v_{j_1 \dots j_r}$ of Δ^I is the intersection of all the n facets in

$$\mathcal{F}(\Delta^I) - \{F_{j_i}^i \mid i = 1, \dots, r\}.$$

In particular, the n facets that intersect at the vertex $v_{0 \dots 0}$ are:

$$\mathcal{F}(\Delta^I) - \{F_0^i \mid i = 1, \dots, r\} = \{F_1^1, \dots, F_{n_1}^1, \dots, F_1^r, \dots, F_{n_r}^r\}$$

And the facets not incident to $v_{0 \dots 0}$ are F_0^1, \dots, F_0^r . Note that for any $1 \leq i \neq i' \leq r$, the intersection of F_0^i and $F_0^{i'}$ is exactly a codimension two face of Δ^I .

Suppose μ is the characteristic function of Q^n on Δ^I and $\pi_{\mu} : Q^n \rightarrow \Delta^I$ is the corresponding quotient map the locally standard action on Q^n . Then according to the preceding discussion, we can cut Q^n along the facial submanifolds $\pi_{\mu}^{-1}(F_0^1), \dots, \pi_{\mu}^{-1}(F_0^r)$ which will gives us a \mathbb{Z}_2 -core V^n of Q^n . The panels of V^n are denoted by P_1, \dots, P_r where P_i consists of 2^{n_i} copies of F_0^i .

Since we assume M^n is connected, by Lemma 2.1, there exists a $\lambda \in \text{Col}_m(V^n)$ such that $M^n \cong M(V^n, \lambda)$ and $m = \text{rank}(\lambda) \leq r = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. In addition, $\mathbb{R}\mathcal{Z}_{\Delta^I} = S^{n_1} \times \cdots \times S^{n_r}$ is a principal $(\mathbb{Z}_2)^r$ -bundle over Q^n .

Let $\iota : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^r$ be the standard inclusion and define $\lambda_0 = \iota \circ \lambda$. So λ_0 is a $(\mathbb{Z}_2)^r$ -coloring on V^n . Obviously, $\text{rank}(\lambda_0) = \text{rank}(\lambda)$. Without loss of generality, we assume $\{\lambda_0(P_1), \dots, \lambda_0(P_m)\}$ is a basis of $L_{\lambda_0} \subset (\mathbb{Z}_2)^r$. Choose $\omega_1, \dots, \omega_{r-m} \in (\mathbb{Z}_2)^r$ so that $\{\lambda_0(P_1), \dots, \lambda_0(P_m), \omega_1, \dots, \omega_{r-m}\}$ forms a basis of $(\mathbb{Z}_2)^r$. Then we define a sequence of coloring $\lambda_1, \dots, \lambda_{r-m} \in \text{Col}_r(V^n)$ as following: for each $1 \leq j \leq r-m$,

$$\lambda_j(P_i) := \begin{cases} \lambda_0(P_i), & 1 \leq i \leq m \text{ or } m+j < i \leq r; \\ \omega_{i-m}, & m+1 \leq i \leq m+j. \end{cases}$$

Then $\text{rank}(\lambda_{j+1}) = \text{rank}(\lambda_j) + 1$ for any $0 \leq j < r-m$. Let $\theta_j : V^n \times (\mathbb{Z}_2)^k \rightarrow M(V^n, \lambda_j)$ be the quotient map of the glue-back construction. Then by the proof of Lemma 2.2 in [2], there exists a sequence of closed connected manifolds:

$$K_{r-m} \xrightarrow{\eta_{r-m}} K_{r-m-1} \xrightarrow{\eta_{r-m-1}} \cdots \xrightarrow{\eta_2} K_1 \xrightarrow{\eta_1} K_0 = M^n, \quad (8)$$

where each $K_j = \theta_j(V^n \times L_{\lambda_j})$ is a connected component of $M(V^n, \lambda_j)$ and the $\eta_j : K_j \rightarrow K_{j-1}$ is a double covering. Notice that $\text{rank}(\lambda_{r-m}) = r$, so $M(V^n, \lambda_{r-m}) = K_{r-m}$ and λ_{r-m} is a maximally independent $(\mathbb{Z}_2)^r$ -coloring on V^n . Then both K_{r-m} and $\mathbb{R}\mathcal{Z}_{\Delta^I}$ are connected principal $(\mathbb{Z}_2)^r$ -bundles over Q^n . Then the Lemma 2.5 in [2] asserts that K_{r-m} must be homeomorphic to $\mathbb{R}\mathcal{Z}_{\Delta^I}$.

To analyze the relationship between the total \mathbb{Z}_2 -cohomology rank of these spaces, we need the following lemma.

Lemma 3.2. *For a closed connected manifold N and any double covering $\xi : \tilde{N} \rightarrow N$, we must have $\text{hrk}(\tilde{N}, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(N, \mathbb{Z}_2)$. And $\text{hrk}(\tilde{N}, \mathbb{Z}_2) = 2 \cdot \text{hrk}(N, \mathbb{Z}_2)$ if and only if ξ is a trivial double covering.*

Proof. The Gysin sequence of $\xi : \tilde{N} \rightarrow N$ in \mathbb{Z}_2 -coefficient reads:

$$\cdots \longrightarrow H^{i-1}(N, \mathbb{Z}_2) \xrightarrow{\phi_{i-1}} H^i(N, \mathbb{Z}_2) \xrightarrow{\xi^*} H^i(\tilde{N}, \mathbb{Z}_2) \longrightarrow H^i(N, \mathbb{Z}_2) \xrightarrow{\phi_i} \cdots$$

where $\phi_i(\gamma) = \gamma \cup e_\xi$, $\forall \gamma \in H^i(N, \mathbb{Z}_2)$ and $e_\xi \in H^1(N, \mathbb{Z}_2)$ is the first Stiefel-Whitney class (Mod 2 Euler class) of ξ . Then by the exactness of the Gysin sequence, we have:

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^i(\tilde{N}, \mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^i(N, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) + \dim_{\mathbb{Z}_2} \ker(\phi_i) \\ &= 2 \cdot \dim_{\mathbb{Z}_2} H^i(N, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_i) \\ &\leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(N, \mathbb{Z}_2) \end{aligned}$$

So $\text{hrk}(\tilde{N}, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(N, \mathbb{Z}_2)$. If $\text{hrk}(\tilde{N}, \mathbb{Z}_2) = 2 \cdot \text{hrk}(N, \mathbb{Z}_2)$, then $\text{Im}(\phi_i) = 0$ for all $i \geq 0$. In this case, we claim the first Stiefel-Whitney class e_ξ must be zero. Indeed, if $e_\xi \in H^1(N, \mathbb{Z}_2)$ is not zero, by the Poincaré duality for N , there must be some element $\alpha \in H^{n-1}(N, \mathbb{Z}_2)$ where n is the dimension of N , so that $\alpha \cup e_\xi$ is the generator of $H^n(N, \mathbb{Z}_2) \cong \mathbb{Z}_2$. This would contradict $\text{Im}(\phi_{n-1}) = 0$. Moreover, since the first Stiefel-Whitney class completely classifies a double covering, so e_ξ is zero will imply that ξ is a trivial double covering. \square

Now, let us come back to the proof of our Theorem 1.5. Since

$$\text{hrk}(K_0, \mathbb{Z}_2) = \text{hrk}(M^n, \mathbb{Z}_2) = 2^m, \quad \text{hrk}(K_{r-m}, \mathbb{Z}_2) = \text{hrk}(\mathbb{R}\mathcal{Z}_{\Delta^I}, \mathbb{Z}_2) = 2^r,$$

so by Lemma 3.2, we must have $\text{hrk}(K_{j+1}, \mathbb{Z}_2) = 2 \cdot \text{hrk}(K_j, \mathbb{Z}_2)$ for any $0 \leq j < r - m$ in the sequence (8). Then by Lemma 3.2 again, each $\eta_j : K_j \rightarrow K_{j-1}$ must be a trivial double covering. But this is not possible since each K_j is connected. Therefore, the only possibility is that $K_0 = K_{r-m}$, i.e. $r = m$ and $M^n \cong \mathbb{R}\mathcal{Z}_{\Delta^I}$. So M^n is homeomorphic to a product of spheres $S^{n_1} \times \cdots \times S^{n_r}$ ($r = m$). \square

Remark 3.3. Most of the results in [2] and this paper have parallel statements for principal real torus bundles over quasitoric manifolds. The ideas are similar, though there are some extra ingredients in the later case.

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